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DESCRIPTION OF BEAM-MATTER INTERACTION IN THE COVARIANCE MATRIX FORMALISM

– Application to Modification of Emittance and Twiss Parameters –

A.-S. Müller

Abstract

Perturbation of a beam by an obstacle in the beam path is a well-known problem. Monitoring devices to determine beam parameters or stripping foils to change the charge state of ions are placed in the beam path and cause energy loss and scattering processes. This note describes these processes in the framework of covariance matrices, which makes no a priori assumption of the actual phase space distribution and is applicable to a variety of processes involving changes in particle position and angle. Exact compact expressions are derived whose series expansions yield the usual approximate expressions for increase in beam emittance and change in Twiss parameters. A numerical example is given for the case of scattering and compared with results derived from a corresponding Monte Carlo calculation.

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1 Introduction

Various effects arising from beam-matter interaction (BMI) can be most conveniently described in the covariance matrix formalism. These phenomena include effects acting on the angle, like multiple Coulomb scattering in material (for example in a stripping foil), and directly or indirectly on the position, like momentum changes in dispersive regions (energy loss in material, also radiation effects like synchrotron radiation or bremsstrahlung, even if that is not beam-matter interaction, strictly speaking). All these different effects have one thing in common: they can be described through probability density functions in the phase plane. The same is true for the beam itself. The so-called "beam matrix" is just a covariance matrix representing the distribution of the individual particles in the phase plane. The elements of this covariance matrix are defined by $C_{ij} = \langle ij \rangle - \langle i \rangle \langle j \rangle$ where i and j represent the two coordinates. For a two-dimensional beam matrix they are position and angle. In the following the coordinates will be denoted by "x" (for position) and "y" (for the angle). The covariance matrix representing the beam particle distribution can also be expressed in terms of Twiss parameters and beam emittance:

$$C \;=\; \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix} \;=\; \begin{pmatrix} \varepsilon_0\beta_0 & -\varepsilon_0\alpha_0 \\ -\varepsilon_0\alpha_0 & \varepsilon_0\gamma_0 \end{pmatrix}. \tag{1}$$

For Gaussian distributions, the Twiss parameters describe an ellipsoidal shape of the phase plane boundary. The emittance can be defined from the determinant of this matrix

$$\varepsilon_{RMS}^2 \overset{\text{def}}{=} \det C \;=\; C_{xx}\, C_{yy} \;-\; C_{xy}^2 \tag{2}$$

with $C_{xy} = C_{yx}$ for symmetry reasons. This can be easily verified with the right hand side of Eq. (1): $\det C \;=\; \varepsilon_0^2 \left(\beta_0\gamma_0 \;-\; \alpha_0^2\right) \;=\; \varepsilon_0^2$. The definition of the RMS emittance holds true for any particle distribution.

Beam-matter interaction acts on the particle distribution and therefore on the emittance. From the above definitions it is clear that a change in beam emittance, meaning in the determinant of C, will also affect the Twiss parameters which can be derived from Eq. (1) (see also Eq. (8) and Eq. (16)). Apart from the direct emittance increase due to immediate changes in position and angle caused by beam-matter interaction, this associated change in Twiss parameters creates an additional emittance increase from the mismatch between original and new Twiss parameters. In most cases this contribution is small.

The effect of the beam-matter interaction on the particle distribution can be described by a convolution of the beam distribution function $\rho(\vec{x})$ with the one representing the distribution generated by the interaction $\phi(\vec{y})$

$$\rho'(\vec{z}) \;=\; \int d^n x\; \rho(\vec{x}) \int d^n y\; \phi(\vec{y})\; \delta^n\left(\vec{z} - \vec{x} - \vec{y}\right). \tag{3}$$

It is a general property of covariance matrices that the covariance matrix of the convoluted distribution C' (particle distribution after BMI) is given by the sum of the initial beam's covariance matrix C and the covariance matrix of the interaction ΔC: $C' = C + \Delta C$. This is shown in detail in Appendix A. In this note the relations for emittance increase and Twiss parameter change are derived in a simple way for multiple Coulomb scattering and energy straggling, making use of this convenient property of covariance matrices.

The methods and results presented in this paper were also used in the studies leading to the proposal of a low-β stripping insertion for lead ions in the PS to SPS transfer line (TT2) [1].

2 Multiple Coulomb Scattering

Multiple Coulomb scattering is treated quite thoroughly in the literature (see for example [2–4]). It offers, however, a very good possibility to demonstrate the usefulness of the covariance matrix formalism.

For simplicity reasons we start with the composition of the matrix for a very thin obstacle (e.g. a stripping foil) where displacements caused by the change in angle are negligible and we need only consider the change in angle itself. The next step will be a full description for obstacles of arbitrary lengths. For all these considerations we will suppose that the angle depends only on the initial energy of the incident particles and no corrections for energy loss need to be made. In any case it can be shown that the RMS scattering angle remains the same [5].

2.1 Thin Scatterer

For a very thin stripper there is no change in position at the exit of the foil, only a change in angle has taken place. Therefore the covariance matrix for a thin scatterer can be written as

$$\Delta C_{\text{thin}} = \begin{pmatrix} \Delta C_{xx} & \Delta C_{xy} \\ \Delta C_{yx} & \Delta C_{yy} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \langle\theta^2\rangle \end{pmatrix} \tag{4}$$

where $\sqrt{\langle\theta^2\rangle}$ is the RMS scattering angle imposed by the process in question. For the multiple Coulomb scattering this angle is given by [6]

$$\sqrt{\langle\,\theta^2\rangle} = 13.6\, z\, \frac{1}{\beta p} \sqrt{\frac{x}{X_0}} \left\{ 1 + 0.038\, \ln \frac{x}{X_0} \right\} \tag{5}$$

where X_0 is the radiation length of the material, z the charge state, p the total momentum (in MeV/c), and β the velocity of the incident particle relative to the speed of light. The thickness of the scattering material is denoted by x. Since the new covariance matrix is given by the sum of the original one and the covariance matrix of the scatterer, the matrix describing the particle distribution after scattering at a thin obstacle is

$$C' = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} + \langle\theta^2\rangle \end{pmatrix}. \tag{6}$$

This yields the following exact expression for the emittance:

$$\begin{aligned}\varepsilon \;=\; \sqrt{\det C'} \;&=\; \sqrt{\varepsilon_0^2 \;+\; C_{xx}\langle\theta^2\rangle} \\ &=\; \varepsilon_0 \left(1 \;+\; \frac{1}{2}\,\frac{C_{xx}}{\varepsilon_0^2}\,\langle\theta^2\rangle \;+\; \text{higher order terms}\right).\end{aligned}$$

Replacing the covariance matrix element C_{xx} by the corresponding representation in beam parameters (see Eq. (1)), this yields the well-known expression for an emittance increase due to multiple scattering in first order (see also [2] or [3]):

$$\boxed{\Delta\varepsilon \;\approx\; \frac{1}{2}\,\beta_0\,\langle\theta^2\rangle.} \tag{7}$$

The new Twiss parameters can be immediately read off the covariance matrix and are

$$\begin{aligned}\alpha \;&=\; \frac{-C'_{xy}}{\sqrt{\det C'}} \;=\; \frac{\varepsilon_0\,\alpha_0}{\varepsilon_0 \;+\; \Delta\varepsilon} \\ \beta \;&=\; \frac{C'_{xx}}{\sqrt{\det C'}} \;=\; \frac{\varepsilon_0\,\beta_0}{\varepsilon_0 \;+\; \Delta\varepsilon} \\ \gamma \;&=\; \frac{C'_{yy}}{\sqrt{\det C'}} \;=\; \frac{\varepsilon_0\,\gamma_0 \;+\; \langle\theta^2\rangle}{\varepsilon_0 \;+\; \Delta\varepsilon}\end{aligned} \tag{8}$$

As the expressions of Eq. (7) show the effective change in Twiss parameters is mainly due to the change in the determinant of the covariance matrix and is therefore closely connected to the change in emittance.

2.2 Obstacle of Arbitrary Thickness

To find the covariance matrix representing an obstacle of finite length L it is convenient to divide this length into n slices of length L/n. Since the length is no longer negligible, we will also have to take into account the displacement X that each particle experiences when traversing the scattering material. With small changes in angle Δ_i for each slice, the displacements at the "entry" of the next slice are

$$\begin{aligned}x_1 \;&=\; \Delta_0\,\frac{L}{n} && \text{at the first slice } (x_0 \overset{!}{=} 0) \\ x_2 \;&=\; x_1 \;+\; (\Delta_0 + \Delta_1)\,\frac{L}{n} && \text{at the second slice} \\ &\ldots\end{aligned}$$

so that the total displacement and angle at the exit of the scatterer are represented by

$$\begin{aligned}\Theta \;&=\; \sum_{i=0}^{n-1} \Delta_i \\ X \;&=\; \frac{L}{n}\sum_{i=0}^{n-1} (n-i)\;\Delta_i\end{aligned} \tag{9}$$

The corresponding covariance matrix for the generated particle distribution at the end of the scatterer is given by

$$\Delta C_L = \begin{pmatrix} \langle X^2\rangle - \langle X\rangle\langle X\rangle & \langle X\Theta\rangle - \langle X\rangle\langle\Theta\rangle \\ \langle \Theta X\rangle - \langle \Theta\rangle\langle X\rangle & \langle\Theta^2\rangle - \langle\Theta\rangle\langle\Theta\rangle \end{pmatrix}$$

where the off-diagonal elements are identical. All we have to do now is to calculate the expectation values keeping in mind that $\langle\Delta_i^2\rangle = \langle\Theta^2\rangle/n$ and $\langle\Theta^2\rangle = \langle\theta^2\rangle$. The expectation value for the angle is obviously

$$\langle\Theta\rangle = \langle\sum_{i=0}^{n-1}\Delta_i\rangle = \sum_{i=0}^{n-1}\langle\Delta_i\rangle = 0. \tag{10}$$

The same holds for $\langle X\rangle$. The expectation value of Θ^2 is given by

$$\langle\Theta^2\rangle = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\langle\Delta_i\Delta_j\rangle = \sum_{i=0}^{n-1}\langle\Delta_i^2\rangle = n\,\frac{\langle\theta^2\rangle}{n} = \langle\theta^2\rangle \tag{11}$$

where we made use of the fact that only terms $i = j$ contribute, since the Δ_i are independent between slices and therefore zero according to Eq. (10). This is also true for for the expectation values of ΘX and X^2:

$$\begin{aligned}\langle\Theta X\rangle &= \frac{L}{n}\sum_{i,j=0}^{n-1}(n-i)\,\langle\Delta_i\Delta_j\rangle = \frac{L}{n}\left[n\sum_{i=0}^{n-1}\langle\Delta_i^2\rangle - \sum_{i=0}^{n-1} i\,\langle\Delta_i^2\rangle\right] \\ &= L\,\langle\theta^2\rangle - \frac{L\,\langle\theta^2\rangle}{n^2}\,\frac{n(n+1)}{2}.\end{aligned}$$

In the limit $n \to \infty$, this becomes

$$\langle\Theta X\rangle = \frac{1}{2}\,L\,\langle\theta^2\rangle. \tag{12}$$

The expectation value of X^2 can be written as

$$\langle X^2\rangle = \frac{L^2}{n^2}\sum_{j=0}^{n-1}(n-j)^2\,\langle\Delta_j^2\rangle$$

since only terms with $i = j$ contribute. Using $\sum_{j=0}^{n-1} j^2 \to n^3/3$ for $n \to \infty$ the expectation value becomes in this limit

$$\langle X^2\rangle = \frac{L^2}{n^3}\,\langle\theta^2\rangle\left[n^3 + \frac{n^3}{3} - 2n\frac{n^2}{2}\right] = \frac{L^2}{3}\,\langle\theta^2\rangle. \tag{13}$$

The covariance matrix representing an obstacle of arbitrary length L taking into account changes in angle as well as lateral displacement is finally

$$\Delta C_L = \begin{pmatrix} \frac{L^2}{3}\langle\theta^2\rangle & \frac{L}{2}\langle\theta^2\rangle \\ \frac{L}{2}\langle\theta^2\rangle & \langle\theta^2\rangle \end{pmatrix}. \tag{14}$$

The emittance increase is obtained from the determinant of the matrix representing the new distribution $C' = C + \Delta C$ in the same way as for Eq. (7). This procedure yields, in first order approximation and neglecting terms in $\langle\theta^2\rangle^2$, the following expression for the emittance change:

$$\boxed{\Delta\varepsilon \;=\; \frac{1}{2}\,\langle\theta^2\rangle\,\left[\beta_0 \;+\; L\alpha_0 \;+\; \frac{L^2}{3}\,\gamma_0\right]} \tag{15}$$

This shows that the increase in emittance is minimal if the scattering takes place at a waist. For the Twiss parameters the following equations result:

$$\begin{aligned} \alpha &= \frac{\varepsilon_0\,\alpha_0 \;-\; \frac{L}{2}\langle\theta^2\rangle}{\varepsilon_0 \;+\; \Delta\varepsilon} \\ \beta &= \frac{\varepsilon_0\,\beta_0 \;+\; \frac{L^2}{3}\langle\theta^2\rangle}{\varepsilon_0 \;+\; \Delta\varepsilon} \\ \gamma &= \frac{\varepsilon_0\,\gamma_0 \;+\; \langle\theta^2\rangle}{\varepsilon_0 \;+\; \Delta\varepsilon} \end{aligned} \tag{16}$$

Since for most practical application in accelerator physics the scatterer thickness is small, displacements generated by the scattering process can be ignored and Eq. (7) is a good approximation of the emittance increase. If the length L of the obstacle is not precisely known and thus there are errors on L and on $\theta(L)$, or if the influence of the initial momentum spread is no longer negligible, uncertainties on emittance and Twiss parameters can be derived easily by standard error propagation.

2.3 A Numerical Application

To give a numerical example, the expressions derived above were used to calculate the modification of emittance and Twiss parameters for two different regimes: one with $\alpha = 0$ and one with $\alpha \neq 0$ but both cases with an initial emittance of 0.49 μm and for an RMS scattering angle of $\theta_{\mathrm{RMS}} = 0.094$ mrad. The case of the upright phase plane ellipse corresponds to the situation at the proposed stripping insertion in the PS-SPS transfer line [1], and the case of the tilted ellipse to conditions at the entry of the TT2 line.
The analytical results are compared with Monte Carlo (MC) simulation in Table 1. For the MC estimation of the parameters, two-dimensional (Gaussian) distributions were generated (8000 particles) with the half axes of the corresponding phase plane ellipse determined from the respective Twiss parameters as described in Appendix B. The scattering process was simulated by folding the angular distribution with the distribution function for the scattering angle. The statistical uncertainties given in Table 1 for the Monte Carlo parameters were calculated using standard error propagation where off diagonal terms were taken into account:

$$\sigma^2(\mathscr{P}) \;=\; \sum_{i,k=1}^{3} \frac{\partial\mathscr{P}}{\partial c'_i}\,\frac{\partial\mathscr{P}}{\partial c'_k}\,\Lambda_{ik}(C') \tag{17}$$

where $\mathscr{P}$ stands for α, β, γ or ε respectively, the c'_i represent the elements of the beam's covariance matrix after scattering, and the elements $\Lambda_{ik}(C')$ of the error matrix are defined by

$$\Lambda(C') = \begin{pmatrix} \langle C_{xx}C_{xx}\rangle - \langle C_{xx}\rangle\langle C_{xx}\rangle & \langle C_{xx}C_{xy}\rangle - \langle C_{xx}\rangle\langle C_{xy}\rangle & \langle C_{xx}C_{yy}\rangle - \langle C_{xx}\rangle\langle C_{yy}\rangle \\ \langle C_{xy}C_{xx}\rangle - \langle C_{xy}\rangle\langle C_{xx}\rangle & \langle C_{xy}C_{xy}\rangle - \langle C_{xy}\rangle\langle C_{xy}\rangle & \langle C_{xy}C_{yy}\rangle - \langle C_{xy}\rangle\langle C_{yy}\rangle \\ \langle C_{yy}C_{xx}\rangle - \langle C_{yy}\rangle\langle C_{xx}\rangle & \langle C_{yy}C_{xy}\rangle - \langle C_{yy}\rangle\langle C_{xy}\rangle & \langle C_{yy}C_{yy}\rangle - \langle C_{yy}\rangle\langle C_{yy}\rangle \end{pmatrix}$$

in the same convention for x and y as usual (meaning that y represents the angle x'). For the expectation value $\langle C_{xx}C_{xx}\rangle$ the expression $\left(\sum_i x_i^4\right)/n^2$ is used as an estimator. The other expectation values of covariance element products are calculated accordingly.

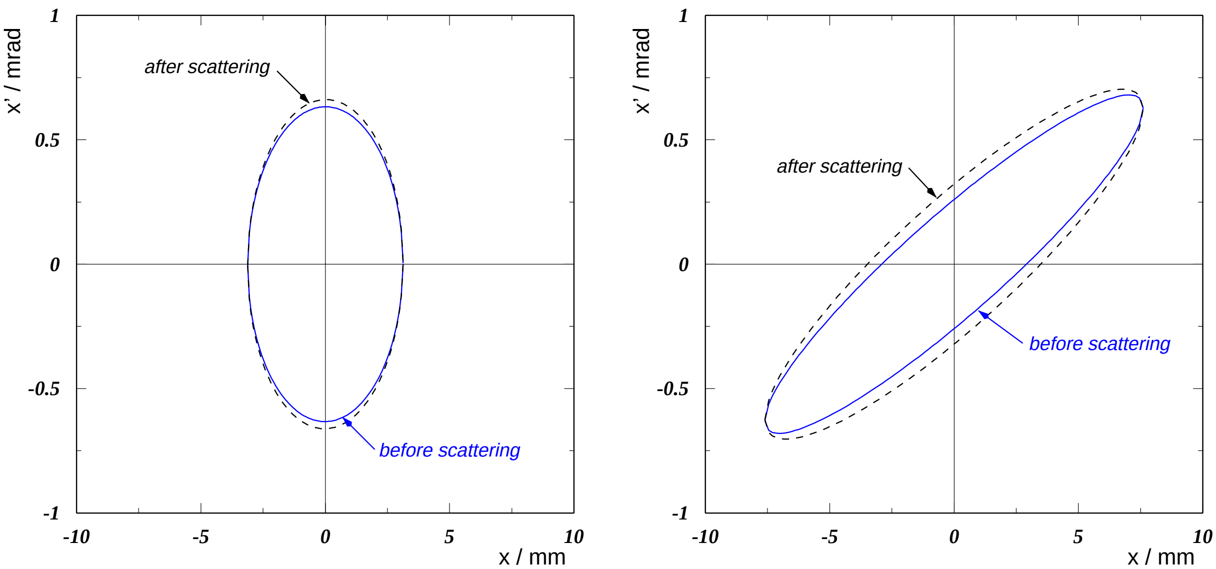

Figure 1: *Change of phase plane ellipse due to a scattering process for $\alpha = 0$ (left) and $\alpha \neq 0$ (right). The initial emittance for both cases is identical. The simultaneous change of emittance and Twiss parameters is clearly visible on the right hand plot. The underlying parameters were determined from the MC generated distribution.*

Analytical and MC results for the modification of Twiss parameters and emittance show a very good agreement. The change of shape of the phase plane ellipse is shown in Fig. 1. The left hand plot corresponds to the left part of Table 1 where $\alpha = 0$, the right hand plot to the $\alpha \neq 0$ case. The solid lines represent the parameters determined from the original distribution before scattering, the dashed lines to the results obtained from the folding of original particle distribution and scattering matrix. In both plots the change in emittance is clearly visible, and in the right plot also the change in Twiss parameters is apparent.

3 Generation of Energy Spread

The generation of additional energy spread is a further case that can be conveniently described in the covariance matrix formalism. The energy spread can for example be generated by the energy loss straggling (energy loss distribution) associated with energy loss by ionisation. In this process each particle loses an individual amount of energy which is distributed around an average energy loss. In the case of a thick scatterer the distribution is Gaussian, whereas for thin scatterers the theories of Landau or Vavilov have to be applied (see [8]). Regardless of the actual distribution, the changes in beam emittance and Twiss parameters are derived in the following.

The physical position of a particle is described as the sum of the contributions from the betatron oscillation and the momentum dependent trajectory. In the absence of a magnetic field, a change in total momentum changes neither physical position nor angle. The betatronic contribution (on which the calculation of the emittance relies) is therefore modified by an amount equivalent to the change in momentum dependent contribution. This leads to an immediate change in calculated beam emittance. The average change in (betatronic) position ("X") and angle ("X′") is of course described by the dispersion and its derivative:

$$\begin{aligned} \langle X \rangle &= \langle D\, \delta_i \rangle \quad \text{and} \quad \langle X' \rangle = \langle D'\, \delta_i \rangle \\ \text{where} \quad \delta_i &= \frac{p_0 - p_i}{p_0} \end{aligned} \tag{18}$$

and p_i, p_0 are the momentum of the individual particle after passage of the obstacle and the average momentum of the incoming beam, respectively. To compose the covariance matrix the expectation values of X^2, XX' and X'^2 have to be calculated:

$$\begin{aligned} \langle XX \rangle &= \langle D^2 \delta_i^2 \rangle - \langle D\delta_i \rangle^2 = D^2 \left(\langle \delta_i^2 \rangle - \langle \delta_i \rangle^2 \right) \\ \langle XX' \rangle &= DD' \left(\langle \delta_i^2 \rangle - \langle \delta_i \rangle^2 \right) \\ \langle X'X' \rangle &= D'^2 \left(\langle \delta_i^2 \rangle - \langle \delta_i \rangle^2 \right) \end{aligned}$$

The corresponding covariance matrix is therefore

$$\Delta C_{\text{momentum}} = \left(\langle \delta_i^2 \rangle - \langle \delta_i \rangle^2 \right) \begin{pmatrix} D^2 & DD' \\ DD' & D'^2 \end{pmatrix}. \tag{19}$$

Again the particle distribution after the energy loss process is described by the sum of the initial matrix and the matrix describing the energy loss contribution $\Delta C_{\text{momentum}}$. Defining $V_\delta = \langle \delta_i^2 \rangle - \langle \delta_i \rangle^2$ the new emittance can be written as usual as

$$\begin{aligned} \varepsilon = \sqrt{\det C'} &= \sqrt{(\varepsilon_0 \beta_0 + D^2 V_\delta)\, (\varepsilon_0 \gamma_0 + D'^2 V_\delta) - (DD' V_\delta - \varepsilon_0 \alpha_0)^2} \\ &= \varepsilon_0 \sqrt{1 + \frac{1}{\varepsilon_0} [\beta_0 D'^2 + \gamma_0 D^2 + 2\alpha_0 DD']\, V_\delta}. \end{aligned}$$

The change in emittance in first order is therefore given by the momentum spread and the dispersion invariant (see also [3] or [9]):

$$\boxed{\Delta\varepsilon \;=\; \frac{1}{2}\left[\beta_0 D'^2 + \gamma_0 D^2 + 2\alpha_0 D D'\right]\; V_\delta} \tag{20}$$

Since $\delta_i = (p_0 - p_i)/p_0$ the variance of the momentum distribution can also be written as

$$V_\delta \;=\; \langle\delta_i^2\rangle - \langle\delta_i\rangle^2 \;=\; \frac{1}{p_0^2}\,\left(\langle p_i^2\rangle - \langle p_i\rangle^2\right) \;=\; \left(\frac{\sigma_p}{p_0}\right)^2.$$

The new Twiss parameters are simply

$$\begin{aligned} \alpha &= \frac{\varepsilon_0\,\alpha_0 \;-\; DD'\,V_\delta}{\varepsilon_0 \;+\; \Delta\varepsilon} \\ \beta &= \frac{\varepsilon_0\,\beta_0 \;+\; D^2\,V_\delta}{\varepsilon_0 \;+\; \Delta\varepsilon} \\ \gamma &= \frac{\varepsilon_0\,\gamma_0 \;+\; D'^2\,V_\delta}{\varepsilon_0 \;+\; \Delta\varepsilon}. \end{aligned} \tag{21}$$

4 Summary

There are many ways to derive the emittance change due to various kinds of beam-matter interaction (see for example [2–4, 9, 10]). The particle distribution point-of-view using the covariance matrix formalism, however, provides a compact method to quantify variations of emittance and the associated changes in Twiss parameters making no *a priori* assumption about the shape of the particle distribution. The corresponding exact expressions for multiple scattering and energy loss straggling have been derived in this note. Series expansion yields the usual approximate expressions. Determination of the RMS errors is straightforward from the calculation of the higher order (up to fourth order) moments of the distribution.

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References

[1] L. Durieu et al. A Low-β Insertion in the PS-SPS Transfer Line to Limit Emittance Blow-Up Due to Stripping of Lead Ions for LHC. CERN/PS 2001-006 (AE), CERN, March 2001.

[2] M. Della Negra. Multiple Coulomb Scattering of Beam Particles in the Phase Space Ellipse Formalism. SLAC/TN 71-1, SLAC, 1971.

[3] P.J. Bryant. Introduction to Transfer Lines and Circular Machines. CERN Yellow Report 84-04, 1984.

[4] A. Maier. Thick Scatterers Seen Through the Twiss Functions. CERN/PS 98-061 (DI), CERN, October 1998.

[5] B. B. Rossi. *High Energy Particles*. New York: Prentice-Hall, 1952.

[6] D.E. Groom et al. Review of Particle Physics. In *Eur. Phys. J. C 15*. 2000.

[7] H. Grote and F. C. Iselin. The MAD Program, User's Reference Manual. SL Note 90-13 (AP) (Rev. 4), CERN, March 1995.

[8] W. R. Leo. *Techniques for Nuclear and Particle Physics Experiments*. Springer-Verlag, 2nd edition, 1994.

[9] G. Arduini et al. Betatron and Dispersion Matching of the TT2/TT10 Transfer Line for the 26 GeV/c Fast Extraction. SL-Note 99-032 MD, 1999.

[10] M. Sands. Emittance Growth from Radiation Fluctuations. SLAC/AP 47, SLAC, 1985.

A Convolution of n-Dimensional Distributions and Covariance Matrices

If a distribution is obtained from the convolution of two other distributions, its covariance matrix is given by the sum of the covariance matrices of the input distributions. This is easy to see for one-dimensional distribution and is commonly known for Gaussians. It is however also true for any folding of n-dimensional distributions, providing that the variance of the original distributions exists. Let $f = f(x_1, x_2, \ldots, x_n)$ be an arbitrary distribution function depending on n variables x_i which is normalised:

$$\int d^n x \; f(\vec{x}) \;=\; 1 \quad \text{where} \quad \int d^n x \;=\; \int \prod_{i=1}^{n} dx_i$$

Define further

$$F_i(x) \;=\; \int \prod_{k \neq i} dx_k \; f(\vec{x}) \qquad \text{and} \tag{22}$$

$$F_{ij}(x) \;=\; \int \prod_{k \neq i,j} dx_k \; f(\vec{x}) \qquad \text{so that} \quad \int dx_j \; F_{ij}(x) \;=\; F_i(x) \tag{23}$$

$$\text{and} \quad \int dx_i \; F_i(x) \;=\; 1. \tag{24}$$

The expectation value of x_i is

$$\langle x_i \rangle \;=\; \int d^n x \; x_i \; f(\vec{x}) \;=\; \int dx_i \; x_i \; F_i(x).$$

Let $g(\vec{y})$ be a distribution function fulfilling the same conditions as $f(\vec{x})$ and $h(\vec{z})$ represent a distribution composed of the two:

$$h(\vec{z}) \;=\; \int d^n x \; f(\vec{x}) \int d^n y \; g(\vec{y}) \; \delta^n \left(\vec{z} - \vec{x} - \vec{y}\right) \tag{25}$$

The expectation value of z_i is given by the sum of the expectation values of x_i and y_i. After integration over all components $\neq i$ this reads

$$\begin{aligned} \langle z_i \rangle \;=\; \int d^n z \; z_i \; h(\vec{z}) \;&=\; \int dz_i \; z_i \int dx_i \int dy_i \; F_i(x) \; G_i(y) \; \delta\left(z_i - x_i - y_i\right) \\ &=\; \int dx_i \; F_i(x) \int dy_i \; G_i(y) \left(x_i + y_i\right) \\ &=\; \langle x_i \rangle + \langle y_i \rangle \end{aligned} \tag{26}$$

The square z_i^2 has an expectation value of

$$\langle z_i^2 \rangle \;=\; \int dx_i \; F_i(x) \int dy_i \; G_i(y) \left(x_i + y_i\right)^2 \;=\; \langle x_i^2 \rangle + \langle y_i^2 \rangle + 2 \langle x_i \rangle \langle y_i \rangle. \tag{27}$$

The expectation values of the mixed terms are finally (with the abbreviations $\delta_i = \delta(z_i - x_i - y_i)$ and $\delta_k = \delta(z_k - x_k - y_k)$), after integration over all components $\neq i$ and $\neq k$

$$\begin{aligned}\langle z_i z_k \rangle &= \int dz_i dz_k \; z_i z_k \int dx_i dx_k \; F_{ik}(x) \int dy_i dy_k \; G_{ik}(y) \; \delta_i \; \delta_k \\ &= \int dx_i dx_k \; F_{ik}(x) \int dy_i dy_k \; G_{ik}(y) \; (x_i + y_i)(x_k + y_k) \\ &= \langle x_i x_k \rangle + \langle x_i \rangle \langle y_k \rangle + \langle x_k \rangle \langle y_i \rangle + \langle y_i y_k \rangle \end{aligned} \quad (28)$$

Putting this together, the variance for the composed distribution reads

$$\langle z_i z_k \rangle - \langle z_i \rangle \langle z_k \rangle \;=\; (\langle x_i x_k \rangle - \langle x_i \rangle \langle x_k \rangle) \;+\; (\langle y_i y_k \rangle - \langle y_i \rangle \langle y_k \rangle) \quad (29)$$

which means that the covariance matrix of the new distribution is given by the sum of the covariance matrices of the input distributions or

$$C'_{ik}(z) \;=\; C_{ik}(x) \;+\; C_{ik}(y). \quad (30)$$

A derivation for the two-dimensional case can be found for example in [10].

B Generating Tilted Phase Space Ellipses

A tilted phase space ellipse represents a distribution where the two variables are jointly distributed. To generate such a distribution using Monte Carlo (MC) the two coordinates are generated separately as independent distributions in a coordinate system parallel to the half axes of the ellipse. The pairs of coordinates thus obtained are rotated back to the "laboratory" system where the ellipse is tilted. The angle φ between the two systems is known to be given by

$$\tan 2\varphi \;=\; \frac{2\,\alpha}{\gamma - \beta}. \quad (31)$$

This expression can be easily derived for example from [6] Eq. (28.37). The use of coordinate systems, rotation angle and half axes is explained by Fig. 2.
In the following it will be shown how the ellipse's half axes needed for the MC generation can be derived from known quantities. In the coordinate system parallel to the half axes the ellipse is described by the following equation:

$$1 \;=\; \frac{x'^2}{a^2} \;+\; \frac{y'^2}{b^2} \quad (32)$$

The well-known rotation relation

$$\begin{pmatrix} x \\ y \end{pmatrix}' \;=\; \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

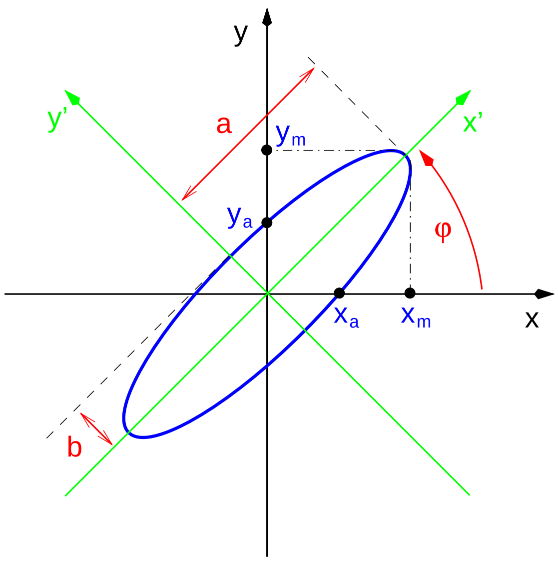

Figure 2: *Sketch showing the two sets of coordinate systems. One (x, y) is the "laboratory" system, the other (x', y') the system parallel to the half axes of the tilted ellipse. The angle between the two systems is φ. The two half axes of the ellipse a and b are indicated. Also shown are the extreme points and the intersections with the axes.*

can be used to express Eq. (32) in the laboratory system:

$$1 \;=\; A\,x^2 \;+\; 2\,B\,xy \;+\; C\,y^2 \;=\; f(x,y) \tag{33}$$

using the definitions

$$A \;=\; \frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}, \quad B \;=\; \cos\varphi\sin\varphi\,\left(\frac{1}{a^2} - \frac{1}{b^2}\right), \quad C \;=\; \frac{\cos^2\varphi}{b^2} + \frac{\sin^2\varphi}{a^2}.$$

The intersection with the axes x_a and y_a are given by $x_a^2 = 1/A$ and $y_a^2 = 1/C$. The extreme points of the ellipse are determined by the conditions $\frac{\partial}{\partial x} f(x,y) = 0$ and $\frac{\partial}{\partial y} f(x,y) = 0$. These conditions lead to linear dependencies of the coordinates $x = -yB/A$ and $y = -xB/C$ which after substitution into Eq. (33) yield the maxima in x and y of the ellipse:

$$x_m^2 \;=\; \frac{C}{AC - B^2} \qquad \text{and} \qquad y_m^2 \;=\; \frac{A}{AC - B^2}.$$

Keeping in mind that the emittance can be expressed as the product of the half axes $\varepsilon = ab$, the symmetric sum $A + C = 1/a^2 + 1/b^2$ can be used to derive a relation for the half axes:

$$s_\pm^2 \;=\; \frac{A+C}{2}\,\varepsilon^2 \;\pm\; \sqrt{\left(\frac{A+C}{2}\,\varepsilon^2\right)^2 - \varepsilon^2}$$

where $s_\pm$ stands for the major half axis a (with the positive sign in front of the square root) and for the minor half axis b (with the negative sign in front of the square root) respectively. Substituting the expressions for A and C and with $\varepsilon = x_m y_a = x_a y_m$, this can be further simplified to

$$s_\pm^2 \;=\; \frac{\varepsilon}{2}\left[(\beta+\gamma) \;\pm\; \sqrt{(\beta+\gamma)^2 \;-\; 4}\right]. \tag{34}$$

	upright ellipse				**tilted ellipse**			
	analytical		*Monte Carlo*		*analytical*		*Monte Carlo*	
	before	after	before	after	before	after	before	after
α	0.00	0.00	-0.004 $\pm$ 0.011	-0.009 $\pm$ 0.011	-2.35	-1.91	-2.42 $\pm$ 0.03	-1.94 $\pm$ 0.02
β / m	4.81	4.61	4.94 $\pm$ 0.05	4.72 $\pm$ 0.05	28.49	23.15	29.28 $\pm$ 0.32	23.64 $\pm$ 0.26
γ / m^{-1}	0.208	0.217	0.203 $\pm$ 0.002	0.212 $\pm$ 0.002	0.230	0.201	0.234 $\pm$ 0.003	0.202 $\pm$ 0.002
ε / μm	0.49	0.51	0.49 $\pm$ 0.01	0.52 $\pm$ 0.01	0.49	0.60	0.49 $\pm$ 0.01	0.61 $\pm$ 0.01

Table 1: *The analytical calculations before scattering are based on* MAD *[7] calculations of the Twiss parameters, for the emittance the input values are given. The effect of scattering has been estimated analytically using the equations given in this paper. The errors given for the Monte Carlo calculations represent the statistical uncertainties for 8000 data points. All calculations are done for a scattering angle of* θ_{RMS} = *0.094 mrad.*